Mathematics 290-A Exam 4 Spring 2007

April 17, 2007

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Name

Technology used: ______________________

Directions:

- **Only write on one side of each page.**
- **Use terminology correctly.**
- **Partial credit is awarded for correct approaches so justify your steps.**

Do Two (2) of these "Computational" Problems

C.1. Without using technology, compute the determinant of the matrix

$$\begin{bmatrix} 0 & -1 & 0 & 1 \\ -2 & 3 & 1 & 6 \\ 1 & -2 & 2 & 3 \\ 0 & 1 & 0 & -2 \end{bmatrix} = 5.$$

C.2. Prove that the function $T : M_{n,n} \to M_{n,n}$ given by $T(A) = A + A^t$ is a linear transformation

C.3. The number $\lambda = 2$ is an eigenvalue of the matrix $\begin{bmatrix} 3 & -2 & 2 \\ -4 & 1 & -2 \\ -5 & 1 & -2 \end{bmatrix}$. Determine a basis for the eigenspace, $E_A(2)$, corresponding to this eigenvalue and state the geometric multiplicity $\gamma_A(2)$ of this eigenvalue.

$A - 2I = \begin{bmatrix} 3-2 & -2 & 2 \\ -4 & 1-2 & -2 \\ -5 & 1 & -2-2 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$ so $E_A(2) = \left\langle \left\{ \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} \right\} \right\rangle$ and $\gamma_A(2) = 1$.

Do Two (2) of these "In text, class or homework" problems

M.1. Prove **two** (2) of the following.

(a) If A is diagonalizable and B is similar to A then B is diagonalizable.

(b) If A is diagonalizable and invertible then A^{-1} is diagonalizable.

(c) Suppose A and B have the same eigenvalues and each eigenvalue has the same algebraic and geometric multiplicity in A as it does in B. If A is diagonalizable, then A and B are similar.

M.2. A square matrix A is **idempotent** if $A^2 = A$. Show that if A is an idempotent matrix then the numbers 0 and 1 are both eigenvalues of A and that they are the only eigenvalues of A.

M.3. Theorem $ILTLI$ (Injective Linear Transformations and Linear Independence) tells us that if $T : U \to V$ is a linear transformation then the image of any linearly independent set is linearly independent. Without using this theorem, prove that if $S = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a linearly independent set in the vector space U and $T : U \to V$ is an injective linear transformation, then $R = \{T(\vec{u}_1), T(\vec{u}_2), T(\vec{u}_3)\}$ is a linearly independent set in the vector space V.

Do two (2) of these "Other" problems

T.1. The set $B = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ is a basis for $\mathbf{C}^2$.Define a function $T : \mathbf{C}^2 \to \mathbf{C}^2$ by: if $\vec{x} = a \begin{bmatrix} 3 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, then $T(\vec{x}) = a \begin{bmatrix} 4 \\ 2 \end{bmatrix} + b \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Use the fact (which you do not have to prove) that T is a linear transformation to find the matrix A that satisfies $T(\vec{x}) = A\vec{x}$ for every vector $\vec{x} \in \mathbf{C}^2$.

T.2. Suppose that A is a 4×4 matrix with exactly two distinct eigenvalues, 6 and -7 and let $E_A(6)$ and $E_A(-7)$ be the respective eigenspaces.

(a) Write all possible characteristic polynomials of A that are consistent with $E_A(6) = 3$

(b) Write all possible characteristic polynomials of A that are consistent with $E_A(-7) = 2$.

T.3. An $n \times n$ matrix A is called **nilpotent** if, for some positive integer k, $A^k = O$, where O is the $n \times n$ zero matrix. Prove that 0 is the only eigenvalue of any nilpotent matrix.